$$\frac{u_m}{u_{m-1}} = \prod_{\alpha\in\mu^\times_{f,m}} g(\alpha) = \prod_{\substack{\text{Gal orbit}\\ \text{of } \alpha}} g(\sigma\alpha) = N_{L'/L}(\alpha)$$

Lemma B $\Rightarrow \frac{u_m}{u_{m-1}^{\varphi}} \in 1+\mathfrak{m}_L^m$

$$\Rightarrow N_{L'/K}(u) = N_{L/K}\left(\frac{u_m}{u_{m-1}}\right) = N_{L/K}\left(\frac{u_m}{u_{m-1}^{\varphi}}\right) \in N_{L/K}(1+\mathfrak{m}_L^m) \subseteq 1+\mathfrak{m}_K^m = U_K^m$$

We still need $\supseteq$.

08.01.2019

It appears that we will omit the rest of the proof.

Notation. $A^G = \{a\in A \mid ga=a \ \forall g\in G\}$ where A is an ab gp; $G \curvearrowright A$

Ex. $L^G = K$, $(L^\times)^G = K^\times$ in Galois theory

Ex. $K:=\mathbb{R}$, $L:=\mathbb{C}$, $G=\{1,\sigma\}$ where σ is the couplex conjugation

$$\{\pm 1\} \hookrightarrow \mathbb{C}^\times \xrightarrow{(\ )^2} \mathbb{C}^\times$$

$$\{\pm 1\} = \{\pm 1\}^G \hookrightarrow (\mathbb{C}^\times)^G = \mathbb{R}^\times \xrightarrow{(\ )^2} \mathbb{R}^\times$$ note that surjectivity is not preserved

In the language of homological algebra: we are dealing with a left exact functor here.

Now we shall recall some notions of homological algebra.

Def. $\mathbb{Z}[G] = \left\{ \sum n_g g \mid n_g\in\mathbb{Z} \right\}$, multiplication induced from G.

Any G-module A is a left $\mathbb{Z}[G]$-module.

Def. Invariants: A^G, coinvariants: $A_G := A/\{ga-a \mid g\in G,\ a\in A\}$

Lemma. $(\ )^G : \mathrm{Mod}_{\mathbb{Z}[G]} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[G]}$ is left exact,

$(\ )_G : \mathrm{Mod}_{\mathbb{Z}[G]} \longrightarrow \mathrm{Mod}_{\mathbb{Z}[G]}$ is right exact.

Def. In an abelian cat $\mathcal{A}$, an object P is projective if

$$\begin{tikzcd} & & & P \arrow[d, "f"] \arrow[dl, dashed, "\exists \tilde f"'] & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

An object I is injective if projective in $\mathcal{A}^{op}$, i.e.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[dl, dashed, "\exists"] & C \arrow[r] & 0 \\ & I & & & \end{tikzcd}$$

holds.

Def. $\mathcal{A}$ has enough injectives if $\forall X \in \mathrm{Ob}\,\mathcal{A}$ $\exists X \hookrightarrow I$ with I injective.

$\mathcal{A}$ has enough projectives if $\forall X \in \mathrm{Ob}\,\mathcal{A}$ $\exists P \twoheadrightarrow X$ with P projective.

Thm. $\mathcal{A}$ abelian cat w/ enough injectives and $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor b/w abelian cats then there are right derived functors R^iF s.t. any s.e.s. $0 \to A \to B \to C \to 0$ gives rise to a l.e.s. $0 \to FA \to FB \to FC \to R^1FA \to \dots$ and this is functorial in short exact sequences.

Computing derived functors

$X \in \mathrm{Ob}\,\mathcal{A}$, $\mathcal{A}$ has enough injectives $\Rightarrow$ $X \hookrightarrow I^0$ for some injective I^0

$\Rightarrow X \hookrightarrow I^0 \twoheadrightarrow X/I^0$, $\quad X/I^0 \hookrightarrow I^1$ injective

$\Rightarrow X \hookrightarrow I^0 \xrightarrow{d^0} I^1$. $\quad \mathrm{Coker}\, d^0 \hookrightarrow I^2$ injective, repeat

$\Rightarrow X \hookrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \dots$ injective resolution $I^\bullet$

Then $R^iF = H^i(F(I^\bullet))$. (This is well-defined, i.e. independent of $I^\bullet$)

Ex. Reality check: $R^0F(X) = F(X)$.

Indeed, $R^0F(X) = H^0(F(I^\bullet)) = \mathrm{Ker}\,(F(I^0) \to F(I^1)) = \mathrm{Im}\,(F(X) \hookrightarrow F(I^0)) = F(X)$.

Ex. For any ring R, Mod_R has enough injectives and projectives.

1) Let M be any R-module. Then $\mathrm{Hom}_R(M,-) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is left exact and has right derived functors $\mathrm{Ext}^i_R(M,-)$

2) $- \otimes_R M$ is right exact and has left derived functors $\mathrm{Tor}^R_i(M,-)$

If N is injective: $\mathrm{Ext}^i_R(M,N) = \begin{cases} \mathrm{Hom}_R(M,N) & i=0 \\ 0 & i \neq 0 \end{cases}$

Rmk. $\mathrm{Hom}_{\mathbb{Z}G}(A,B)$ has left G-action $(gf)(x) = g f(g^{-1}x)$, and $A^G \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},A)$.

Pf: $f(1) \longleftrightarrow (f : \mathbb{Z} \to A)$. is an iso. $\square$

Since this iso is functorial in A, there is an equivalence of functors $(-)^G = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},-)$ and similarly $(-)_G = \mathbb{Z} \otimes_{\mathbb{Z}G} -$

Note that $\mathbb{Z}$ here is endowed with a trivial G-action, i.e. $gn = n$ $\forall g \in G$ $\forall n \in \mathbb{Z}$.

Def. G group, A a G-module (i.e a $\mathbb{Z}G$-module). Then the group cohomology of G with coeffs in A is $H^i(G;A) := (R^i(-)^G)(A) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z},A)$ and the group homology of G is $H_i(G;A) := (L_i(-)_G)(A) = \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z},A)$

Rmk. $H^i(G,A) = H^i(BG,A)$ where BG is the classifying space.

Let $f: G' \to G$ be a homomorphism of groups. Then any G-module A has a canonical G'-module structure: $g'\cdot a := f(g')\cdot a$

This gives rise to a functor $\mathrm{Mod}_{\mathbb{Z}G} \to \mathrm{Mod}_{\mathbb{Z}G'}$

Ex. $A^G \leq A^{G'}$ is a subgroup, follows from def.

This injection is functorial in A, thus we obtain a natural transformation of functors $(-)^G \Rightarrow (-)^{G'}$, which in turn induces $R^i(-)^G \Rightarrow R^i(-)^{G'}$, so for any G-module A we obtain $H^i(G,A) \to H^i(G',A)$. (#)

Application. 1) $H \leq G$ subgp. The induced maps of (#) in this case are called restrictions: $\mathrm{Res}^G_H: H^i(G,A) \to H^i(H,A)$

2) $H \trianglelefteq G$. Then A^H is a G/H-module.

$$\begin{tikzcd} H^i(G/H; A^H) \arrow[r] \arrow[dr, "\mathrm{Inf}"'] & H^i(G; A^H) \arrow[d, "A^H \subseteq A"] \\ & H^i(G;A) \end{tikzcd}$$

$\mathrm{Inf}: H^i(G/H; A^H) \to H^i(G;A)$ is called the inflation.

The horizontal map is induced by $G \to G/H$.

10.01.20

Thm. $H \trianglelefteq G$, A a G-module. Then there is a spectral sequence
$$E_2^{p,q} := H^p(G/H; H^q(H;A)) \Longrightarrow H^{p+q}(G;A)$$
Hochschild–Serre spectral sequence.

Pf: Observe $(-)^G = ((-)^H)^{G/H}$ is an equality of functors, so $(-)^G$ is the composition of the two left exact functors $(-)^H$ and $(-)^{G/H}$.

Grothendieck sp. sq. ($\approx$ chain rule for right derived functors) yields the HS sp. sq (in fact the latter is a special case of the former).

Thm. $E_\bullet^{\bullet,\bullet}$ sp. sq then there is a so-called 5-term sequence
$$0 \to E_2^{1,0} \to H^1(\text{limit}) \to E_2^{0,1} \to E_2^{2,0} \to H^2(\text{limit})$$

Cor. $0 \to H^1(G/H; A^H) \xrightarrow{\mathrm{Inf}} H^1(G;A) \xrightarrow{\mathrm{Res}} H^1(G;A)^{G/H} \to H^2(G/H; A^H) \to H^2(G;A)$

Rmk. This can be obtained in a completely direct way, see Sheet 10.

Warning. Common misconception: if $G \curvearrowright A$ is trivial, i.e. $A^G = A$ then $H^i(G;A)=0 \;\forall i \geq 1$ is false.

Ex. $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, $A := \{\pm 1\}$ $\quad \{\pm 1\} \hookrightarrow \mathbb{C}^\times \xrightarrow{(\,)^2} \mathbb{C}^\times$ but $\{\pm 1\} \hookrightarrow \mathbb{R}^\times \underset{\text{not surj}}{\xrightarrow{(\,)^2}} \mathbb{R}^\times \to \underbrace{H^1(G;\{\pm 1\})}_{\neq 0}$

Thm. Let A, B be G-modules. Then $\mathrm{Ext}^i_{\mathbb{Z}G}(B,A)$ can be computed in the following two ways:

1) $0 \to A \hookrightarrow I^0 \to \dots$ inj res, $\mathrm{Ext}^i_{\mathbb{Z}G}(B,A) = H^i(\mathrm{Hom}_{\mathbb{Z}G}(B, I^\bullet))$

2) $\dots \to P_0 \twoheadrightarrow B \to 0$ proj res, $\mathrm{Ext}^i_{\mathbb{Z}G}(B,A) = H^i(\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, A))$

Pf: 1) is just the def., 2): [Weibel] □

For any $r \geq 0$, let P_r be the free $\mathbb{Z}G$-module with basis $(r+1)$-tuple $(g_0, \dots, g_r)$.
Then endow P_r with a $\mathbb{Z}G$-module structure: $g(g_0, \dots, g_r) = (gg_0, \dots, gg_r)$
and let $d_r : P_r \longrightarrow P_{r-1}$, $d_r(g_0, \dots, g_r) = \sum_{j=0}^{r} (-1)^j (g_0, \dots, \hat{g_j}, \dots, g_r)$

Thm. $\dots \to P_1 \to P_0 \to \mathbb{Z} \to 0$ is a proj res of $\mathbb{Z}$ as a $\mathbb{Z}G$-module.

Pf: Sheet 9, $\dots \to \mathbb{Z}[G\times G] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ free resolution. □

Combining the two Thms:

$\mathrm{Ext}^i_{\mathbb{Z}G}(B,A) = H^i(\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, A))$, for this $P_\bullet$ yields

$$H^i(G,A) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z},A) \qquad \mathrm{Hom}_{\mathbb{Z}G}(P_r, A) = \{\varphi : G^{r+1} \to A \mid \varphi(gg_0, \dots, gg_r) = g\varphi(g_0, \dots, g_r)\} =: \tilde{C}^r$$
$$= H^i(\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, A))$$
$$= H^i(\tilde{C}^\bullet)$$

complex of homogeneous chains

Def. G gp, A G-module, $\varphi : G \to A$, $\varphi(\sigma\tau) = \sigma\varphi(\tau) + \varphi(\sigma)$ $\forall \sigma, \tau \in G$. Then this φ is a crossed homomorphism.

Def. $a \in A$: $\sigma \mapsto \sigma a - a$ is a cr. hom, called a principal crossed homomorphism.

Prop. $H^1(G;A) \cong \dfrac{\text{crossed homs}}{\text{pr. crossed homs}}$

Pf: Unravel $\tilde{C}^\bullet$.

Inhomogeneous chains $C^\bullet$, $\exists$ quasi-iso $C^\bullet \xrightarrow{\sim} \tilde{C}^\bullet$ (PO)

See [Milne]. □

Cor. If $G \curvearrowright A$ is trivial then $H^1(G;A) \cong \mathrm{Hom}(G;A)$ (as groups) □

This looks interesting: our goal is to learn things about the Galois gp, and we have $H^1(G; \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(G; \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(\underbrace{G/[G;G]}_{\text{abelianisation}}, \mathbb{Q}/\mathbb{Z})$

Def. Induced G-module: any G-module iso to $\mathbb{Z}[G] \otimes_{\mathbb{Z}} X$ for some ab gp X.

Coinduced G-module: any G-module iso to $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], X)$ for some ab. gp. X.

Lemma. A coinduced $\Rightarrow H^i(G; A) = 0 \quad \forall i \geq 1$,

A induced $\Rightarrow H_i(G; A) = 0 \quad \forall i \geq 1$.

Pf: Suppose A is induced; $N \otimes_{\mathbb{Z}G} A = N \otimes_{\mathbb{Z}G} \left(\mathbb{Z}[G] \otimes_{\mathbb{Z}} X \right) \simeq N \otimes_{\mathbb{Z}} X$

This is functorial in N. $\Rightarrow (- \otimes_{\mathbb{Z}G} A) = (- \otimes_{\mathbb{Z}} X)$ as functors

$$\Rightarrow \mathrm{Tor}_i^{\mathbb{Z}G}(N, A) = \mathrm{Tor}_i^{\mathbb{Z}}(N, X)$$

Set $N := \mathbb{Z}$, which is flat, hence $\mathrm{Tor} = 0$.

Similar for coinduced A.

Let K be a field, L/K fin Galois extension, $G = \mathrm{Gal}(L/K)$.

Thm. (Hilbert 90) $H^1(G, L^\times) = 0$. (This formulation is due to Noether.)

Thm. (Additive H90) $H^i(G, L) = 0 \quad \forall i \geq 1$.

Pf of H90: Let $\varphi: G \to L^\times$ be a crossed homomorphism. Wts φ is principal.

For $a \in L^\times$ let $b := \sum_{\sigma \in G} \varphi(\sigma) \cdot \sigma(a)$

Caveat: since we are working in $L^\times$, we use multiplicative notation instead of the usual additive one.

If $b \neq 0$ then $\tau b = \sum_\sigma \tau\varphi(\sigma) \cdot \tau\sigma(a) = \sum_\sigma \varphi(\tau)^{-1} \varphi(\tau\sigma) \cdot \tau\sigma(a) = \varphi(\tau)^{-1} b$

$$\Rightarrow \varphi(\tau) = \frac{b}{\tau b} = \frac{\tau(b^{-1})}{b^{-1}}, \quad \text{i.e. } \varphi \text{ is principal.}$$

Dedekind linear independence of characters $\Rightarrow \exists a$ s.t. $b \neq 0$.

Exc. Why is $H^1(G, \mathcal{O}_K^\times)$ hopeless? (K a number field or a local field)

Pf of AH90: Normal Basis Theorem: $\exists \alpha \in L$ s.t. $\{\sigma_1\alpha, \dots, \sigma_n\alpha\}$ is a basis of L as a K-vector space, $G = \{\sigma_1, \dots, \sigma_n\}$.

Then $K[G] \xrightarrow{\sim} L$, $\sigma \longmapsto \sigma\alpha$ is an iso. of G-modules.

$\Rightarrow L \simeq_{\mathrm{Mod}_{\mathbb{Z}G}} \mathbb{Z}[G] \otimes_{\mathbb{Z}} K$, ($L$ is induced, hence the vanishing of the homology)

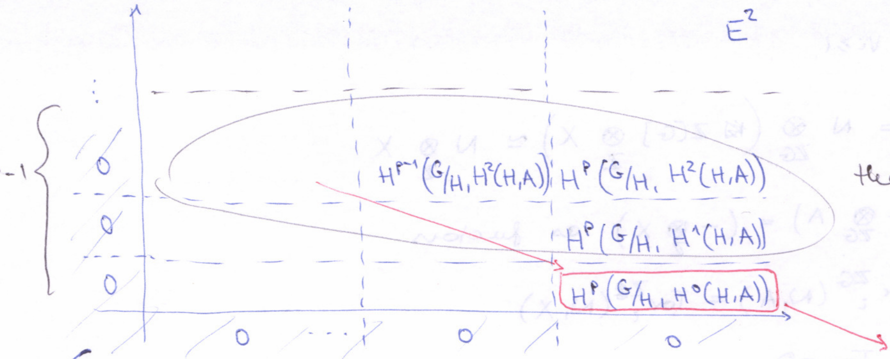
$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], K) = \mathrm{Hom}_K(\underbrace{\mathbb{Z}G \otimes_{\mathbb{Z}} K}_{K[G] \simeq L}, K)$, hence L is coinduced $\Rightarrow$ vanishing of H^i

Note: we may have switched from left to right action here.

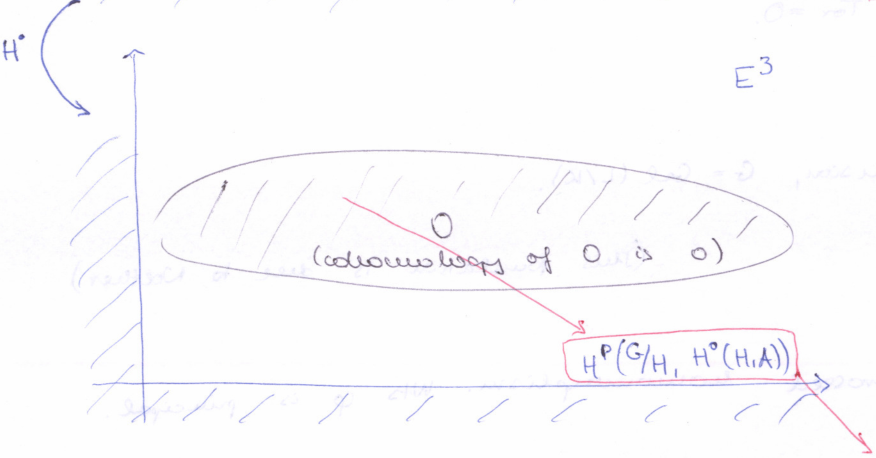
Prop. $H \triangleleft G$, A a G-module, $H^i(H,A) = 0$ $\forall\, 0<i<r$.

Then $0 \to H^r(G/H, A^H) \xrightarrow{\text{Inf}} H^r(G,A) \xrightarrow{\text{Res}} H^r(H,A)$. (See also Sheet 10.4)

Pf: Use Hochschild – Serre. $E_2^{p,q} = H^p(G/H, H^q(H,A)) \Rightarrow H^{p+q}(G,A)$



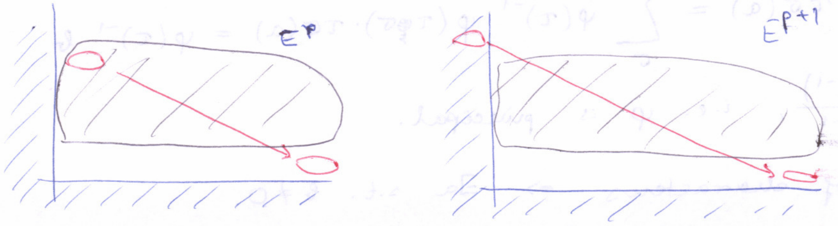
these are 0 since $H^i(H,A) = 0$ for $i = 1, \dots, p-1$

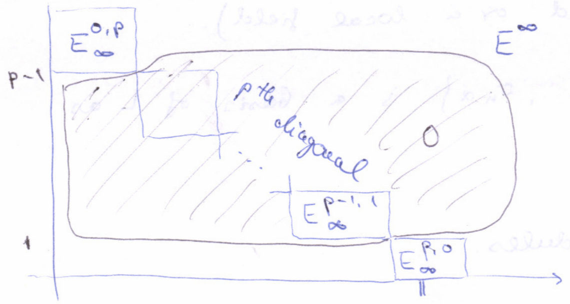


The interesting thing is what happens when the differential comes from above the black zone.

This ~~happens~~ doesn't happen yet for $d_p^{p,0} : E_p^{0,p-1} \longrightarrow E_p^{p,0}$

But for higher values, d comes from the blue zeros, hence nothing changes.



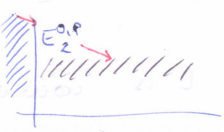


$H^p(G/H, H^0(H,A))$

Convergence: $H^p(G,A)$ has a ~~limit~~ filtration whose graded pieces are the $E_\infty^{i,p-i}$

$$H^p(G/H, A^H) \hookrightarrow H^p(G,A) \longrightarrow E_\infty^{0,p}$$

$E_2^{0,p} = H^0(G/H, H^p(H,A))$



Hence $H_2^{0,p}$ may only get replaced by a subgp in higher pages.

But since we only want to show that there is a surjection, this suffices. $\square$

Addendum to the proof of AH90, p.55:

Lemma. If G is finite then
$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], X) \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}} X$$
$$\varphi \longmapsto \sum_{g\in G} g \otimes \varphi(g^{-1})$$
is an iso of G-modules. ($\sum_{g\in G}$: this makes sense since $\#G<\infty$)

The idea behind this: $M \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], X) = \mathrm{Hom}_{\mathbb{Z}}(\bigoplus_g \mathbb{Z}, X) \underset{\substack{\uparrow \\ \text{we use finiteness here}}}{\simeq} \bigoplus_g \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, X) \simeq \mathbb{Z}[G] \otimes_{\mathbb{Z}} X$

Note. The terms "induced" and "coinduced" have been swapped sometime in the course of history.

Def. G gp, $H \le G$, M a $\mathbb{Z}[H]$-module.
$$\mathrm{Ind}_H^G(M) := \{\varphi: G \to M \mid \varphi(gh) = h\varphi(g) \ \forall h\in H \ \forall g\in G\} \quad \text{induction}$$
We endow $\mathrm{Ind}_H^G(M)$ with a G-module structure:
$$(\varphi+\varphi')(g) := \varphi(g) + \varphi'(g)$$
$$(g\cdot\varphi)(x) := \varphi(xg)$$

Shapiro's Lemma. G gp, $H \le G$, M $\mathbb{Z}[H]$-module. Then there is a canonical iso
$$H^r(G, \mathrm{Ind}_H^G(M)) \xrightarrow{\sim} H^r(H, M)$$

We need to make some preparations before we turn to the proof.

Lemma A a) For every G-module M, H-module N there is an iso
$$\mathrm{Hom}_{\mathbb{Z}G}(M, \mathrm{Ind}_H^G(N)) \simeq \mathrm{Hom}_{\mathbb{Z}H}(M, N)$$
b) The functor $\mathrm{Ind}: \mathrm{Mod}_{\mathbb{Z}H} \to \mathrm{Mod}_{\mathbb{Z}G}$ is exact,

c) and preserves injectivity of objects.

PF (SKETCH): $\varphi: M \to \mathrm{Ind}_H^G(N)$ given. Define $\beta: M \to N$, by $\beta(m) = \varphi(m)(1_G)$, and check that this gives the iso in a).

b) + c): [Milne: CFT, p. 56]. Or for c), use that Ind_H^G is right adjoint to Res_H^G.

PF OF SL: For $r=0$: $M^H = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}, M) \underset{\text{Lemma A a)}}{\simeq} \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, \mathrm{Ind}_H^G(M)) = \mathrm{Ind}_H^G(M)^G$ ✓

Let $r \ge 1$, and $M \to I^\bullet$ be an injective resolution. Then $\mathrm{Ind}_H^G(M) \to \mathrm{Ind}_H^G(I^\bullet)$ is an injective resolution by Lemma A b) + c).

$H^r(G, \mathrm{Ind}_H^G(M)) = H^r((\mathrm{Ind}_H^G(I^\bullet))^G)$ by the construction of derived functors

group cohomology ↗ ↑ "abstract" cohomology

$\simeq H^r((I^\bullet)^H)$ by the argument for $r=0$, we have entry-wise isos $\mathrm{Ind}_H^G(I^\ell)^G \simeq (I^\ell)^H$

$= H^r(H, M)$ by construction of derived functors □

Rmk. One can use Shapiro's Lemma to give an alternative proof that coinduced modules have vanishing higher cohomology.

Def. G gp, $H \leq G$, $|G:H| < \infty$, S a (finite) set of coset representatives for G/H in G, M a G-module. Define $\mathrm{Nm}_{G/H}(m) := \sum_{s \in S} sm$ for $m \in M$, called the norm.

Note that this induces a map $M^H \longrightarrow M^G$.

Rmk. This generalises the notion of norm and trace:

Ex. L/K fin Galois, $G := \mathrm{Gal}(L/K)$, $M := L^\times$

Then $\mathrm{Nm}_{G/1}(x) = \prod_{\sigma \in G} \sigma x = N_{L/K}(x)$ for $x \in L^\times$

Ex. L/K fin Galois, $M := (L, +)$

Then $\mathrm{Nm}_{G/1}(x) = \sum_{\sigma \in G} \sigma x = \mathrm{Tr}_{L/K}(x)$ for $x \in L$.

First extension. This map lifts to higher cohomology groups.

Def. Corestriction: $\mathrm{Cor}_H^G : H^r(H, M) \longrightarrow H^r(G, M)$ defined as follows:

1) $\mathrm{Ind}_H^G(M) \longrightarrow M$

$\varphi \longmapsto \sum_{s \in S} s\varphi(s^{-1})$

2) SL gives $H^r(H, M) \xrightarrow{\sim} H^r(G, \mathrm{Ind}_H^G(M))$

Cor_H^G is the composition $H^r(H,M) \xrightarrow[2)]{\sim} H^r(G, \mathrm{Ind}_H^G(M)) \xrightarrow{1)} H^r(G, M)$

Exc. For $r = 0$, this agrees with $\mathrm{Nm}_{G/H}$.

Prop. $H \leq G$, $|G:H| < \infty$. Then $\mathrm{Cor}_H^G \circ \mathrm{Res}_H^G : H^r(G,M) \longrightarrow H^r(G,M)$ is multiplication by $|G:H|$.

Pf: $H^r(G,M) \xrightarrow{\mathrm{Res}_H^G} H^r(H,M) \xrightarrow{\mathrm{Cor}_H^G} H^r(G,M)$

$\varphi \longmapsto \sum_s s\varphi(s^{-1}) = \underbrace{|G:H|}_{\# S} \cdot 1 \cdot \varphi(1)$

↑ these are all the same □

Cor. 1. If $\#G < \infty$ then $\#G \cdot H^r(G,M) = 0 \quad \forall r \geq 1$

Cor. 2. If $\#G < \infty$ and M is fin. gen. as an abelian gp then $H^r(G,M)$ is fin. for $r \geq 1$.

PF OF 1: Use Prop. for $H = \{1\}$. We have $|G:H| = \#G < \infty$, so it follows that

$$H^r(G,M) \longrightarrow \underbrace{H^r(H,M)}_{=0 \text{ for } r\geq 1} \longrightarrow H^r(G,M) \quad \text{is multiplication by } \#G$$

This argument is called standard transfer argument.

PF OF 2: $H^r(G,M) = H^r(\tilde{C}^\bullet)$

$f \in \tilde{C}^\ell$ is represented by a map $G^{\times(r+1)} \longrightarrow M$, thus $\tilde{C}^\ell$ is fin gen as an abelian group as a map is determined by what it does to generators.

Hence the cohomology groups of $\tilde{C}^\bullet$ are finitely generated.

$\Rightarrow H^r(G,M)$ is fin gen as an abelian gp $\forall r \in \mathbb{Z}$

Now use Cor. 1. $\Rightarrow H^r(G,M)$ is a finite torsion gp $\forall r \geq 1$

We could even give a (useless) upper bound for $\# H^r(G,M)$, growing exponentially.

Cor. 3. Let $\#G < \infty$, G_p a p-Sylow group. Then the map

$$\mathrm{Res}_G^{G_p} : H^r(G,M) \longrightarrow H^r(G_p, M)$$

is injective on the p-primary part.

PF: By the Prop, $\mathrm{Cor} \circ \mathrm{Res}$ is multiplication by $|G:G_p|$, but this is coprime to p, hence invertible mod p. $\Rightarrow \mathrm{Res}_G^{G_p}$ is injective on the p-primary part.

If G is any group, $\varphi : \mathbb{Z}[G] \longrightarrow \mathbb{Z}$, $\sum_g n_g g \longmapsto \sum_g n_g$ is a morphism of G-modules and of rings. (exc.)

Def. $I_G := \mathrm{Ker}\,\varphi$ is called the augmentation ideal in $\mathbb{Z}[G]$.

Exc. I_G is generated by $\{g - e_G \mid g \in G\}$.

Extended Snake Lemma. Given a commutative diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

where the rows are exact we have

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} f \arrow[r] & \operatorname{Ker} a \arrow[r] \arrow[d, "\ker a"] & \operatorname{Ker} b \arrow[r] \arrow[d, "\ker b"] & \operatorname{Ker} c \arrow[d, "\ker c"] & \\ & & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ & 0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] & \\ & & \operatorname{Coker} a \arrow[r] & \operatorname{Coker} b \arrow[r] & \operatorname{Coker} c \arrow[r] & \operatorname{Coker} g' \arrow[r] & 0 \end{tikzcd}$$

where the red part is exact.

Pf: Variation of the proof of the std. Snake Lemma.

Pf 2:

$$E_1^{p,q} := \begin{bmatrix} A \to B \to C \\ A' \to B' \to C' \end{bmatrix} \qquad \tilde{E}_1^{p,q} = \begin{bmatrix} A & B & C \\ \downarrow & \downarrow & \downarrow \\ A' & B' & C' \end{bmatrix}$$

$$E_2^{p,q} := \begin{bmatrix} \operatorname{Ker} f & 0 & 0 \\ 0 & 0 & \operatorname{Coker} g' \end{bmatrix} \qquad \tilde{E}_2^{p,q} = \begin{bmatrix} \operatorname{Ker} f & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & \operatorname{Coker} g' \end{bmatrix}$$

Treat these as double complexes, use spectral sequence magic. □

Tate cohomology

$\# G < \infty$, M G-module, $Nm_G : M \longrightarrow M$, $m \longmapsto \sum_{g \in G} gm$ **norm**

Observe that $\operatorname{Im}(Nm_G) \subseteq M^G$ and $I_G M \subseteq \operatorname{Ker}(Nm_G)$.

$$g' \cdot \sum_g gm = \sum_g (g'g) m \qquad \sum_g g \underbrace{(h-1)m}_{\substack{\text{these} \\ \text{generate } I_G M}} = \sum_g ghm - \sum_g gm = 0$$

We obtain an exact sequence

$$0 \to \operatorname{Ker}(Nm_G)/I_G M \longrightarrow \boxed{M/I_G M \xrightarrow{Nm} M^G} \longrightarrow M^G/Nm_G(M) \longrightarrow 0$$

Suppose $0 \to M' \to M \to M'' \to 0$. Recall: $H_0(G,M) = M/\langle (g-1)m,\ g \in G,\ m \in M\rangle$, $H^0(G,-) = (-)^G$

$$\begin{tikzcd}
\cdots \arrow[r] & H_1(G,M'') \arrow[r] & H_0(G,M') \arrow[r] \arrow[d, "Nm_G"] & H_0(G,M) \arrow[r] \arrow[d, "Nm_G"] & H_0(G,M'') \arrow[r] \arrow[d, "Nm_G"] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & H^0(G,M') \arrow[r] & H^0(G,M) \arrow[r] & H^0(G,M'') \arrow[r] & H^1(G,M') \arrow[r] & H^1(G,M) \arrow[r] & \cdots
\end{tikzcd}$$

Neukirch sets up Tate cohomology using doubly infinite resolutions, but this never became canon in homological algebra.

Def. Tate cohomology: G finite group, M G-module, $r \in \mathbb{Z}$

$$\hat{H}^r(G,M) := \begin{cases} H^r(G,M) & r \geq 1 \\ M^G / Nm_G(M) & r = 0 \\ \operatorname{Ker}(Nm_G) / I_G M & r = -1 \\ H_{-r-1}(G,M) & r \leq -2 \end{cases}$$

17.01.2019

Intermezzo. How does gp cohomology fit into the framework of alg. top. / alg. geo.?

ALGEBRAIC TOPOLOGY

compact connected orientable complex manifold

coherent sheaves

real topology

singular cohomology

$H^n_{sing}(X,A) = \underbrace{H^n(X,\underline{A})}$

sheaf cohomology

loc const sheaf with values in A

ALGEBRAIC GEOMETRY

smooth proper variety

coherent sheaves ($\mathcal{O}_X$-modules)

Zariski topology

$H_{sing}(X,A) = ?$ not so clear def.

or: use sheaf cohomology in the Zariski topology

better: sheaves in the étale topology

$\longrightarrow$ étale cohomology $H^n_{ét}(X,\mathcal{F})$

This has all the properties one would expect from alg. top., e.g. Künneth formula.

Def. K field, G_K absolute Galois group. The Galois cohomology is

$$H^n(K,M) := \underset{\substack{\text{Inf}\\ L/K \text{ finite}}}{\varinjlim} H^n(\mathrm{Gal}(L/K), M^{G_L}) \quad \text{where } M \text{ is a continuous } G_K\text{-module.}$$

Alternatively: define derived functors in the cat. of continuous modules.

Ex. $H^1(G,A) = \mathrm{Hom}(G,A) = \mathrm{Hom}(\hat{\mathbb{Z}}, A)$

$H^1(K,A) = \mathrm{Hom}_{\text{cont.}}(G,A)$

Analogy between $\boxed{\text{Galois cohomology}}$ and $\boxed{\text{étale cohomology}}$

K field	$X := \mathrm{Spec}\, K$

$$\{\text{continuous } G_K\text{-modules}\} \xrightarrow[j]{\sim} \{\text{étale sheaves on } X\}$$

is an equivalence of categories

	$H^n(K,F) \quad \simeq$	$H^n_{ét}(X, jF)$
	$H^n(K, K^{\mathrm{sep},\times})$	$H^n_{ét}(X, \mathbb{G}_m)$
Hilbert 90:	$H^1(K, K^{\mathrm{sep},\times}) = 0$	$H^1_{ét}(X,\mathbb{G}_m) \simeq \mathrm{Pic}(X) \simeq H^1_{Zar}(X,\mathbb{G}_m)$
	L/K finite ext. $\quad L \mid K \quad G_L \subseteq G_K$	$X_L := \mathrm{Spec}\, L \xrightarrow{\pi} X_K := \mathrm{Spec}\, K$
	Res (restriction)	π^* (pullback)
	Ind (induction)	π_* (pushforward)
	Res $\dashv$ Ind	$\pi^* \dashv \pi_*$
Shapiro's Lemma:	$H^n\left(G_K, \mathrm{Ind}_{G_L}^{G_K}(M)\right) \xrightarrow{\sim} H^n(G_L, M)$	Leray spectral sequence
	Cor (corestriction)	transfer

Lemma.[PO] $0 \to M' \to M \to M'' \to 0$ exact sequence of G-modules induces

$$\cdots \to \hat{H}^n(G,M') \to \hat{H}^n(G,M) \to \hat{H}^n(G,M'') \to \cdots$$

Thm. a) If M is an induced (or coinduced) module then $\hat{H}^r(G,M) = 0$, $\forall r \in \mathbb{Z}$.

b) If M is (co)induced, $H \leq G$ subgroup $\Rightarrow \hat{H}^r(H,M) = 0 \quad \forall r \in \mathbb{Z}$

Pf: a) We already know: M induced $\Rightarrow H_r(G,M)=0 \ \forall r\geq 1$

M coinduced $\Rightarrow H^r(G,M)=0 \ \forall r\geq 1$

We have $\#G<\infty$.

Lecture 24: being coinduced is equivalent to being induced,

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],X) \xrightarrow{\sim} \mathbb{Z}[G]\otimes_{\mathbb{Z}} X \quad \text{for an abelian group } X.$$

Thus $\hat{H}^r(G,M)=0$ follows for $r\geq 1$ and $r\leq -2$.

For the remaining r's, check by hand, q.v. [Milne]

Ex. $H_1(G,\mathbb{Z})=G^{ab}$, $H^1(G,\mathbb{Q}/\mathbb{Z})=\mathrm{Hom}(G,\mathbb{Q}/\mathbb{Z})$

b) If M is induced for G, it's also induced for H:

$$M\simeq \mathbb{Z}[G]\otimes_{\mathbb{Z}} X = \bigoplus_{g\in G} g(e\otimes X) = \bigoplus_{h\in H} h\underbrace{\Big(\bigoplus_{s\in G/H} s(e\otimes X)\Big)}_{=\tilde{X}} = \bigoplus_{h\in H} h\otimes \tilde{X} = \mathbb{Z}[H]\otimes_{\mathbb{Z}}\tilde{X}$$

Now apply a).

Lemma. Let G be finite. Then

1) $\hat{H}^r(G,\mathbb{Q})=0 \quad \forall r\in\mathbb{Z}$

2) $\hat{H}^0(G,\mathbb{Z})=\mathbb{Z}/(\#G)\mathbb{Z}$, $\hat{H}^1(G,\mathbb{Z})=0$

3) $\hat{H}^1(G,\mathbb{Q}/\mathbb{Z})=\mathrm{Hom}(G,\mathbb{Q}/\mathbb{Z})\simeq \hat{H}^2(G,\mathbb{Z})$

Pf: 1) Let $m\in\mathbb{Q}^\times$.

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "\cdot m"] \arrow[rr, bend right, "id"'] & \mathbb{Q} \arrow[r, "\cdot \frac{1}{m}"] & \mathbb{Q} \end{tikzcd} \quad \text{induces} \quad \begin{tikzcd} \hat{H}^r(G,\mathbb{Q}) \arrow[r, "\cdot m"] \arrow[rr, bend right, "id"'] & \hat{H}^r(G,\mathbb{Q}) \arrow[r] & \hat{H}^r(G,\mathbb{Q}) \end{tikzcd}$$

Thus multiplication by m on $\hat{H}^r(G,\mathbb{Q})$ is invertible

For $r\geq 1$, we have seen $(\#G)\cdot\hat{H}^r(G,\mathbb{Q})=0$

$\Rightarrow \hat{H}^r(G,\mathbb{Q})=0$ for $r\geq 1$. The case $r<1$ is omitted.

2) $\hat{H}^0(G,\mathbb{Z})=\dfrac{\mathbb{Z}^G}{\mathrm{Nm}(\mathbb{Z})}=\dfrac{\mathbb{Z}}{(\#G)\cdot\mathbb{Z}}$

$\hat{H}^1(G,\mathbb{Z})=H^1(G,\mathbb{Z})=\mathrm{Hom}(G,\mathbb{Z})=0$ since $\mathbb{Z}$ has no nontrivial torsion and G is finite.

3) New technique: dimension shifting.

Let $0\to M'\to M\to M''\to 0$ be a ses of G-modules s.t. $\hat{H}^r(G,M)=0 \ \forall r$

Then $\ldots \xrightarrow{\sim} \hat{H}^r(G,M')\to\hat{H}^r(G,M)\to \boxed{\hat{H}^r(G,M'')\xrightarrow{\sim}\hat{H}^{r+1}(G,M')}\to\ldots$

Apply this to $0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}\to 0$, get $\hat{H}^r(G,\mathbb{Q}/\mathbb{Z})\simeq\hat{H}^{r+1}(G,\mathbb{Z})$.

(aka Vanishing Thm.)

Cool Thm. G finite group, M a G-module. If $\forall H \leq G$: $\hat{H}^1(H,M)=0$ & $\hat{H}^2(H,M)=0$

then $\forall r \in \mathbb{Z}$: $\hat{H}^r(G,M)=0$.

(In fact, $\forall H \leq G$ $\forall r \in \mathbb{Z}$: $\hat{H}^r(H,M)=0$.)

Pf: Step 1. Suppose G is cyclic. Then the Lemma settles the Thm.

Lemma. G cyclic $\Rightarrow$ $\hat{H}^r(G,M) \xrightarrow{\sim} \hat{H}^{r+2}(G,M)$ $\forall r \in \mathbb{Z}$ (2-periodicity of Tate cohomology)

PF IDEA 1: Use proj resolution from exc. sheet 9:

$\to \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{\sum g^i} \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \to \mathbb{Z} \to 0$, note that this is 2-periodic.

PF IDEA 2: [Milne: CFT].

Step 2. Suppose G is solvable. We do induction on $\#G$.

For $\#G < 3$, G is cyclic $\to$ Step 1.

G solvable $\Rightarrow$ $\exists H \trianglelefteq G$, $H \neq G$: G/H is cyclic.

$$0 \to \hat{H}^r(G/H, M) \xrightarrow{\text{Inf}} \hat{H}^r(G,M) \xrightarrow{\text{Res}} \hat{H}^r(H,M) \quad \text{exact sequence}$$

H is solvable and $\#H < \#G$ $\Rightarrow$ $\hat{H}^r(H,M)=0$ by induction

$\Rightarrow \hat{H}^r(G,M) \simeq \hat{H}^r(G/H, M) = 0$ by Step 1. $\forall r \geq 1$

Case $r=0$: verification by hand (using norm).

} proven for $\boxed{\text{solvable}}$ groups and $\boxed{r \geq 0}$

22.01.2019

Step 3. Suppose G is solvable, $r<0$.

$$0 \to I_G \to \mathbb{Z}[G] \xrightarrow{\deg} \mathbb{Z} \to 0 \quad \text{augmentation exact sequence}$$

$$\sum n_g g \mapsto \sum n_g$$

Tensor with M over $\mathbb{Z}$:

$$0 \to M' \to \underbrace{\mathbb{Z}[G] \otimes_{\mathbb{Z}} M}_{\text{induced module}} \xrightarrow{\alpha} M \to 0 \quad \text{where } M' := \operatorname{Ker}(\alpha)$$

induced module $\Rightarrow$ $\hat{H}^r(H, \mathbb{Z}[G] \otimes M) = 0$ $\forall r$

LES $\Rightarrow$ $\hat{H}^r(H,M) \simeq \hat{H}^{r+1}(H,M')$ (#)

$\hat{H}^0(H,M) = \hat{H}^1(H,M) = 0$ because of solvability and Step 2. $\Rightarrow$ $\hat{H}^1(H,M') = \hat{H}^2(H,M')=0$

Now (M', H) satisfy the assumptions of the Thm, and thus we may use the parts already proven $\Rightarrow$ $\hat{H}^r(H,M')=0$ $\forall r$

Use (#) for $r=-1$: $\hat{H}^{-1}(H,M) \simeq \hat{H}^0(H,M)=0$

Do induction decreasing in r $\Rightarrow$ proves the Thm. for G $\boxed{\text{solvable}, r<0}$

Recall. $G_p \le G$ a p-Sylow group, we have already shown $H^p(G,M) \xrightarrow[(*)]{\mathrm{Res}^G_{G_p}} H^p(G_p, M)$ is injective on the p-primary torsion part.

Step 4. Let G be arbitrary.

Known: $\#G \cdot \hat{H}^r(G,M) = 0$

$\Rightarrow \hat{H}^r(G,M)$ is a torsion abelian group

Torsion abelian gps decompose into the direct sum of the primary torsion parts:

$$\hat{H}^r(G,M) = \bigoplus_p \hat{H}^r(G,M)[p^\infty]$$

$(*) \Rightarrow \hat{H}^r(G,M) \xrightarrow{\mathrm{Res}} \hat{H}(G_p, M)$ inj on p-primary torsion, but since p-groups are solvable, $\hat{H}^r(G_p, M) = 0$ by Steps 2 & 3.

Rmk. For $p \to \infty$ on primes, $n_p \to \infty$ there are fin. gen. projective $\mathbb{Z}[G_p]$-modules M_p s.t. $M_p^{\oplus n_p}$ is free but $M_p^{\oplus n}$ is not $\forall n \le n_p$. This means that the classification of such modules is difficult.

Ex. In case frightening aliens happen to kidnap us and force us to compute $H^\bullet(D_{2n}, M)$, use Hochschild–Serre for $0 \to C_n \to D_{2n} \to C_2 \to 0$ and the fact that C_n and C_2 have 2-periodic cohomology.

Addendum to the Proof: to obtain the statement for all subgroups, just use the Thm. for H in place of G.

Thm. (Tate–Nakayama) $\#G < \infty$, C a G-module, $\forall H \le G$:

(A1) $H^1(H,C) = 0$

(A2) $H^2(H,C)$ is cyclic of order $\#H$.

Then if γ is a generator of $H^2(G,C)$ then there is an iso, depending only on the choice of γ: $\hat{H}^r(G,\mathbb{Z}) \xrightarrow{\sim} \hat{H}^{r+2}(G,C)$.

Input Thm. If $K/\mathbb{Q}_p$ is finite then $\forall L/K$ finite Galois, $G := \mathrm{Gal}(L/K)$, $C := L^\times$ satisfy (A1) + (A2).

Combine Tate–Nakayama and IT: $\hat{H}^r(G,\mathbb{Z}) \xrightarrow{\sim} \hat{H}^{r+2}(G, L^\times)$

$r := -2$: $\hat{H}^{-2}(G,\mathbb{Z}) \simeq \hat{H}^0(G, L^\times) \overset{\mathrm{def}}{=} (L^\times)^G / \mathrm{Nm}_{L/K}(L^\times) = K^\times / \mathrm{Nm}_{L/K}(L^\times)$ $\Big\}$ yields $K^\times \longrightarrow G^{ab}$

$\|{\mathrm{def}}$

$H_1(G,\mathbb{Z}) \simeq G^{ab}$ by PS #11

Remember the main task of CFT: $K^\times \longrightarrow \mathrm{Gal}(K^{ab}/K)$

We constructed such a map by LT theory. One can show that this agrees with the map we just obtained. Note that IT is the only place where number thy is involved, everything else is general machinery $\Rightarrow$ can be used for global fields too.

PF OF TATE–NAKAYAMA: Pick $\gamma \in H^2(G, C)$ generator.

Claim. $\forall H \leq G$: $\langle \mathrm{Res}^G_H(\gamma) \rangle = H^2(H, C)$.

Pf: Since $\langle \gamma \rangle = H^2(G, C)$: $\quad n \cdot \gamma = 0$ for $n = \# G$ and $n \cdot \gamma \neq 0$ $\forall n = 1, \dots, \# G - 1$

$$\mathrm{Cor}^G_H \circ \mathrm{Res}^G_H = \text{multiplication by } \frac{\# G}{\# H}$$

$\lightning$ So if $\mathrm{Res}^G_H(\gamma)$ is not a generator in $H^2(H, C)$, $\exists m < \# H$: $m \, \mathrm{Res}^G_H(\gamma) = 0$

$\Rightarrow$ $m \, \mathrm{Cor}\, \mathrm{Res}\,(\gamma) = m \cdot \frac{\# G}{\# H} \cdot \gamma = 0 \Rightarrow \exists \tilde{n}$ s.t. $\tilde{n} \gamma = 0$, $\tilde{n} < \# G$ $\lightning$ $\boxed{c}$

Let φ be a cocycle representative of $H^2(G, C)$, i.e. $\varphi : G \times G \longrightarrow C$ in the inhomogeneous chain complex $(C^\bullet(C), \partial_C)$ s.t. $\partial_C \varphi = 0$ (cocycle condition).

(Recall: $H^2(G, C) = \mathrm{Ext}^2_{\mathbb{Z}G}(\mathbb{Z}, C)$)

splitting module

Define a G-module $C(\varphi) := C \oplus \bigoplus_{g \in G \setminus \{e\}} \mathbb{Z}$. Write x_σ for the $\mathbb{Z}$-basis of the 2nd summand.

with left $\mathbb{Z}G$-structure:
1) $\sigma x_\tau := x_{\sigma\tau} - x_\sigma + \varphi(\sigma, \tau)$
2) the given $\mathbb{Z}G$-structure on C
3) if we hit x_e, take $\varphi(e, e)$ instead.

(This $C(\varphi)$ resembles an induced module.)

Check that this gives a well-def'd $\mathbb{Z}G$-structure on $C(\varphi)$ (direct computation, omitted).

Claim. $H^1(H, C(\varphi)) = 0 = H^2(H, C(\varphi))$ $\forall H \leq G$. (This will imply vanishing in all degrees.)

Pf: Augmentation sequence: $0 \longrightarrow I_G \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$.

Since $\mathbb{Z}[G] = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}$ is induced, $\hat{H}^r(H, \mathbb{Z}[G]) = 0$, so dimension shifting applies:

$\mathbb{Z}/(\# H)\mathbb{Z} \cong H^0(H, \mathbb{Z}) \xrightarrow{\sim} H^1(H, I_G)$ and $\underbrace{H^1(H, \mathbb{Z})}_{= 0, \text{ shown in lec. 25}} \xrightarrow{\sim} H^2(H, I_G)$

There is a map $\alpha : C(\varphi) \longrightarrow \mathbb{Z}[G]$

$c \in C \longmapsto 0$

$x_\sigma \longmapsto \sigma - 1$

(check that this is a morphism of $\mathbb{Z}G$-modules)

Get an exact sequence $0 \longrightarrow C \xrightarrow{\beta} C(\varphi) \xrightarrow{\alpha} I_G \longrightarrow 0$

$[\gamma] \longmapsto 0$

$\Rightarrow$ get les in Tate coh.: $\dots \rightarrow \underbrace{\hat{H}^1(H, C)}_{0 \text{ (A1)}} \rightarrow \hat{H}^1(H, C(\varphi)) \rightarrow \hat{H}^1(H, I_G) \rightarrow \underbrace{\hat{H}^2(H, C)}_{\simeq C_{\# H} \text{ (A2)}} \xrightarrow{\beta} \hat{H}^2(H, C(\varphi)) \rightarrow \underbrace{\hat{H}^2(H, I_G)}_{0 \text{ (see above)}} \rightarrow$

Check that in $C(\varphi)$, the img of $\varphi : G \times G \rightarrow C$ becomes a coboundary: verify by direct computation that $\tilde{\varphi} : G \longrightarrow C(\varphi)$ satisfies $\partial_C(\tilde{\varphi}) = \varphi$

$\sigma \longmapsto x_\sigma$

$\hat{H}^2(H, C) = \langle \mathrm{Res}^G_H(\gamma) \rangle \Rightarrow H^2(\beta)$ is the zero map since the generator γ is sent to 0.

Get: $0 \to \hat{H}^1(H, C(\varphi)) \to \hat{H}^1(H, I_G) \xrightarrow{\varepsilon} \hat{H}^2(H, C) \to 0$ (in Ab)

But $\# \hat{H}^1(H, I_G) = \# H = \# \hat{H}^2(H, C)$ $\Rightarrow$ ε is not only epi but iso $\Rightarrow \hat{H}^1(H, C(\varphi)) = 0$

$\Rightarrow \hat{H}^2(H, C(\varphi)) = 0$ by the les

Use the Vanishing Thm. (aka Cool Thm.) $\Rightarrow \forall H \leq G: \hat{H}^r(H, C(\varphi)) = 0 \; \forall r \in \mathbb{Z}$

$0 \to C \to C(\varphi) \to \boxed{I_G \to 0}$ and $\boxed{0 \to I_G} \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ exact in G-mod

$\Rightarrow 0 \to C \to \underbrace{C(\varphi)}_{\substack{\hat{H}^\bullet(H, C(\varphi)) = 0 \\ \text{by VT}}} \to \underbrace{\mathbb{Z}[G]}_{\substack{\hat{H}^\bullet(H, \mathbb{Z}[G]) = 0 \\ \text{by inducedness}}} \to \mathbb{Z} \to 0$

$\Rightarrow \hat{H}^r(H, \mathbb{Z}) \simeq \hat{H}^{r+2}(H, C)$ from associated les

Rmk. This used to be done using splitting semisimple algebras, hence the name 'splitting module'.

Recall that for finite cyclic groups, $\hat{H}^r(G, M)$ is 2-periodic. 24.01.2019

Def. For G cyclic, define the Herbrand quotient $h(M) := \dfrac{\# \hat{H}^0(G, M)}{\# \hat{H}^1(G, M)}$ whenever these cohomology groups are finite.

Lemma. $\# G < \infty$ cyclic and assume all $h(-)$ are defined.

a) If $0 \to M' \to M \to M'' \to 0$ then $h(M) = h(M') \cdot h(M'')$.

b) If $\# M < \infty$ then $h(M) = 1$.

Pf: a) The ses induces a les:

$$\hat{H}^0(G, M') \to \hat{H}^0(G, M) \to \hat{H}^0(G, M'') \to \hat{H}^1(G, M') \to \hat{H}(G, M) \to \hat{H}^1(G, M'')$$

"Exact hexagon" for the cohomology of cyclic groups.

Recall Lemma X. If $0 \to A' \to A \to A'' \to 0$ are fin ab gps then $\# A = \# A' \cdot \# A''$.

Splice the exact hexagon into pieces and apply Lemma X to all these. $\Rightarrow$ a)

b) Observe that $0 \to M^G \to M \xrightarrow{g-1} M \to M_G \to 0$ where $\langle g \rangle = G$, and

$0 \to \hat{H}^{-1}(G, M) \to M_G \xrightarrow{Nm} M^G \to \hat{H}^0(G, M) \to 0$ are exact sequences

$$\Rightarrow \frac{\# M^G}{\# M} = \frac{\# M}{\# M_G} = 1 \;\&\; \frac{\# \hat{H}^{-1}(G, M)}{\# M_G} = \frac{\# M^G}{\# \hat{H}^0(G, M)} = 1.$$

Apply now $\hat{H}^{-1}(G, M) \simeq \hat{H}^1(G, M)$, this yields the assertion.

Rmk.* In the pf of Tate–Nakayama, we have produced an exact sequence

$$0 \longrightarrow C \longrightarrow C(\varphi) \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

This sequence defines, using Yoneda's interpretation that Ext-groups classify extensions, a $\pi \in \mathrm{Ext}^2(\mathbb{Z}, C)$.

If $r \geq 1$: $\hat{H}^r(G,\mathbb{Z}) = H^r(G,\mathbb{Z}) = \mathrm{Ext}^r_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$, there is a Yoneda product

$$\mathrm{Ext}^p(B,C) \times \mathrm{Ext}^q(A,B) \longrightarrow \mathrm{Ext}^{p+q}(A,C)$$

$$\mathrm{Ext}^2_{\mathbb{Z}G}(\mathbb{Z},C) \times \mathrm{Ext}^r_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) \longrightarrow \mathrm{Ext}^{r+2}_{\mathbb{Z}G}(\mathbb{Z},C).$$

Recall the yet-to-be-proven

Imput Thm. Let $K/\mathbb{Q}_p$ be a finite extension, L/K finite Galois extension, $C := L^\times$. (this is a $G = \mathrm{Gal}(L/K)$-module). This satisfies axioms

(A1) $\hat{H}^1(H,C) = 0$, (A2) $\hat{H}^2(H,C)$ is cyclic of order $\# H$

for all $H \leq G$ subgroups.

Pf: (A1): $\hat{H}^1(H,L^\times) = 0$ is Hilbert 90 in the multiplicative version. ✓

(A2): Wts $\hat{H}^2(H,L^\times)$ is cyclic of order $\# H$.

Case 1. L/K is unramified.

Recall $0 \to \mathcal{O}_L^\times \longrightarrow L^\times \xrightarrow{v_L} \mathbb{Z} \longrightarrow 0$

Claim. $\hat{H}^r(G, \mathcal{O}_L^\times) = 0 \quad \forall r$.

Pf: L/K ur $\Rightarrow$ any uniformiser π_K of K is also a uniformiser of L

$\Rightarrow$ we get a splitting in the category of G-modules: $\hat{H}^r(G,L^\times) \simeq \hat{H}^r(G,\mathcal{O}_L^\times) \oplus \hat{H}^r(G,\mathbb{Z})$

For $r=1$: $\hat{H}^1(G,L^\times) = 1$ by H90 $\to \hat{H}^1(G,\mathcal{O}_L^\times) = 0$. ✓

For $r=0$: $\hat{H}^0(G,L^\times) = \dfrac{(\mathcal{O}_L^\times)^G}{Nm(\mathcal{O}_L^\times)} = \dfrac{\mathcal{O}_K^\times}{Nm(\mathcal{O}_L^\times)}$. Therefore $\hat{H}^0(G,L^\times) = 0 \Leftrightarrow Nm$ is surjective on the unit group $\mathcal{O}_L^\times$.

Lemma 1. $\hat{H}^r(G, k_L^\times) = 0. \quad \forall r \in \mathbb{Z}$

Pf: $r=1$: H90 $\Rightarrow \hat{H}^1(G,k^\times) = 0$.

Since G is cyclic, $\hat{H}^0$ is periodic, $h(k_L^\times) = 1 \underset{\text{def}}{=} \dfrac{\# \hat{H}^0(G,k_L^\times)}{1} \to \hat{H}^0(G,k_L^\times) = 0$ ($\# k_L^\times < \infty$)

$\Rightarrow \hat{H}^r(G,k_L^\times) = 0 \ \forall r$ by periodicity.

Lemma 2. $\hat{H}^r(G,(k_L,+)) = 0 \quad \forall r \in \mathbb{Z}$.

$\ldots \subseteq U^2 \subseteq U^1 \subseteq \mathcal{O}_L^\times \to \hat{H}^r(G,\mathcal{O}_L^\times) \simeq \hat{H}^r(G,U^m) \ \forall m \geq 1 \Rightarrow \hat{H}^r(G,\mathcal{O}_L^\times) = 0$

by a topological argument in (Milne).

$1 \to U^1 \to U \to k_L^\times \to 1$ induces:

$\cdots \to \hat{H}^r(G, U^1) \to \hat{H}^r(G, U) \to \underbrace{\hat{H}^r(G, k_L^\times)}_{0} \to \cdots$

$1 \to U^{m+1} \to U^m \to (k_L, +)$

$\Rightarrow \hat{H}^r(G, O_L^\times) = 0$

$1 \to O_L^\times \to L^\times \xrightarrow{v_L} \mathbb{Z} \to 0 \quad \overset{\text{les}}{\Rightarrow} \quad \hat{H}^r(G, L^\times) \underset{v_L}{\xrightarrow{\sim}} \hat{H}^r(G, \mathbb{Z})$

$1 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 1 \quad \overset{\text{les}}{\Rightarrow} \quad \hat{H}^r(G, \mathbb{Q}/\mathbb{Z}) \to \hat{H}^r(G, \mathbb{Z})$

$\hat{H}^2(G, L^\times) \xrightarrow{\sim} \hat{H}^2(G, \mathbb{Z}) \xleftarrow{\sim} \hat{H}^1(G, \mathbb{Q}/\mathbb{Z}) = \hat{H}^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$

L/K ur $\Rightarrow$ $G = \mathrm{Gal}(L/K)$ cyclic and $(x \mapsto x^{\# k_K})$ is a canonical generator

We have a map $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$

$f \mapsto f(\mathrm{Frob})$

In total, we get $\mathrm{inv}_{L/K} : H^2(G, L^\times) \to \mathbb{Q}/\mathbb{Z}$ Hasse invariant

Thm. a) There exists a unique iso $\mathrm{inv}_K : H^2(\mathrm{Gal}(K^{ur}/K), K^{ur,\times}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$

s.t. $\forall L/K$ fin ur this agrees with $\mathrm{inv}_{L/K}$ along

$$\begin{tikzcd}
H^2(\mathrm{Gal}(L/K), L^\times) \arrow[r, "\sim", "\mathrm{inv}_{L/K}"'] \arrow[d, "\mathrm{Inf}"] & \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z} \arrow[d, hook] \\
H^2(\mathrm{Gal}(K^{ur}/K), K^{ur,\times}) \arrow[r, "\sim"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

b)
$$\begin{tikzcd}
H^2(\mathrm{Gal}(K^{ur}/K), K^{ur,\times}) \arrow[r, "\mathrm{Res}"] \arrow[d, "\mathrm{inv}_K"] & H^2(\mathrm{Gal}(L^{ur}/L), L^{ur,\times}) \arrow[d, "\mathrm{inv}_L"] \\
\mathbb{Q}/\mathbb{Z} \arrow[r] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

for every finite (not nec. ur) extension of K.

This finishes the proof in the unramified case.

Case 2. L/K arbitrary.

Recall: If $H \triangleleft G$, $H^i(H, M) = 0 \ \forall 0 < i < r \Rightarrow 0 \to H^r(G/H, M^H) \xrightarrow{\mathrm{Inf}} H^r(G, M) \xrightarrow{\mathrm{Res}} H^r(H, M)$ (#)

H90 $\Rightarrow H^1(H, L^\times) = 0$, we may apply (#) with $M := L^\times$, $r = 2$, H is the subgp corresponding to L

$\Rightarrow 0 \to H^2(G/H, L^\times) \xrightarrow{\mathrm{Inf}} H^2(G, L^\times) \xrightarrow{\mathrm{Res}} H^2(H, L^\times)$

Thm. $K/\mathbb{Q}_p$ fin ext $\Rightarrow$ $\exists$ canonical iso $\mathrm{inv}_K : H^2(K, K^{sep,\times}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$,

still called the Hasse invariant, s.t. for L/K finite:

$$\begin{array}{ccccccc} 0 \longrightarrow & H^2(G/H, L^\times) & \longrightarrow & H^2(K, K^{sep,\times}) & \xrightarrow{\mathrm{Res}} & H^2(K, K^{sep,\times}) & \\ & & & \downarrow \mathrm{inv}_K \text{ (the new one)} & & \downarrow \mathrm{inv}_K \text{ (the new one)} & n = [L:K] \\ 0 \longrightarrow & \frac{1}{n}\mathbb{Z}/\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \end{array}$$

and this defines an iso

$$\mathrm{inv}_{L/K} : H^2(\mathrm{Gal}(L/K), L^\times) \xrightarrow{\sim} \tfrac{1}{n}\mathbb{Z}/\mathbb{Z} \qquad \text{where } G = \mathrm{Gal}(K^{sep}/K),\ H = \mathrm{Gal}(K^{sep}/L)$$

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Recall: we are in the process of proving IT.

In the unramified case:

Prop. 1 L/K ur fin $\Rightarrow$ $\hat{H}^r(G, U) = 0$ $\forall r \in \mathbb{Z}$ where $U = \mathcal{O}_L^\times$

Proven using H90 and decompositions (higher unit filtration).

Def. Invariant map on the finite level:

$$0 \to U_L \hookrightarrow L^\times \xrightarrow{v_L} \mathbb{Z} \to 0$$
$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \qquad \text{exact sequences of } G\text{-modules}$$

$\Rightarrow$ get connecting homs $\hat{H}^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow[\sim]{\partial} \hat{H}^2(G, \mathbb{Z})$ iso since $\mathbb{Q}$ has trivi coh.

and $H^2(G, L^\times) \xrightarrow[\sim]{v_L} H^2(G, \mathbb{Z})$ iso by Prop 1. Here $G = \mathrm{Gal}(L/K)$.

The invariant map is the composition.

Def. $H^2(L/K) := \varinjlim_{\substack{\mathrm{Inf} \\ \tilde{L}/K \text{ fin Gal} \\ \text{subextn}}} H^2(\mathrm{Gal}(\tilde{L}/K), \tilde{L}^\times)$

Rmk. If L/K fin Gal then $H^2(L/K) \simeq \hat{H}^2(\mathrm{Gal}(L/K), L^\times)$, and if $L = K^{sep}$ then

$$H^2(K^{sep}/K) = \underbrace{H^2(K, K^{sep,\times})}_{\text{Galois cohomology}}$$

Def. (continued) L/K Galois fin ur. $\Rightarrow$ G cyclic, canonically gen'd by the lift of the Frob ($x \mapsto x^{\# G}$)

$$H^2(L/K) \xrightarrow{\sim} \hat{H}^2(G, \mathbb{Z}) \xrightarrow{\sim} \hat{H}^1(G, \mathbb{Q}/\mathbb{Z}) \underset{\text{def}}{\xrightarrow{\sim}} H^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow[\substack{\text{eval at} \\ \text{Frob lift}}]{} \mathbb{Q}/\mathbb{Z}$$

(The img will be contained in $\frac{1}{\# G}\mathbb{Z}/\mathbb{Z}$.) This is $\mathrm{inv}_{L/K} : H^2(L/K) \to \mathbb{Q}/\mathbb{Z}$.

Thm. There is a unique iso $\mathrm{inv}_K : H^2(K^{ur}/K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ s.t. $\forall L/K$ fin ur:

inv_K induces the iso $\mathrm{inv}_{L/K} : H^2(L/K) \xrightarrow{\sim} \frac{1}{[L:K]}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$

$$\left.\begin{array}{l} K^{ur} \\ \quad | \ \text{max ur} \\ L \\ \quad | \ \text{fin} \ \} \ \mathrm{inv}_{L/K} \\ K \end{array}\right\} \mathrm{inv}_K$$

LCFT, lecture 28–29

<u>Prop.</u> L/K fin ~~ur~~ ext, $[L:K]=n$ then

$$\begin{tikzcd} H^2(K^{ur}/K) \arrow[r, "Res"] \arrow[d, "\sim"', "inv_K"] \arrow[dr, phantom, "\circlearrowright"] & H^2(L^{ur}/L) \arrow[d, "\sim"', "inv_L"] \\ \mathbb{Q}/\mathbb{Z} \arrow[r, "\cdot n"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

$$\begin{tikzcd} L^{ur} = K^{ur} \arrow[d, dash] & \text{max ur} \\ L \arrow[d, dash, "fin"] & \\ K & \end{tikzcd}$$

[Milne T1.7, L1.8] for details: no big ideas, just compatibility statements.

This finishes the construction of the inv map in the unramified case.

<u>Rmk.</u> For K a field: $H^2(K^{sep}/K) = Br(K)$ Brauer group.

Now we need to handle the general case, where ramification enters the game.

<u>Thm.</u> ([Milne Thm. 2.1]) Let $K/\mathbb{Q}_p$ be fin. Then there is a canonical iso $inv_K : H^2(K^{sep}/K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ such that if $[L:K]=n<\infty$ then

$$\begin{tikzcd} 0 \arrow[r] & H^2(L/K) \arrow[r] & H^2(K^{sep}/K) \arrow[r, "Res"] \arrow[d, "\sim"', "inv_K"] \arrow[dr, phantom, "G"] & H^2(K^{sep}/L) \arrow[d, "\sim"', "inv_L"] \\ 0 \arrow[r] & \frac{1}{n}\mathbb{Z}/\mathbb{Z} \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r, "\cdot n"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

Pf: <u>Claim A.</u> L/K Galois, $[L:K]=n<\infty$ $\rightarrow$ $H^2(L/K)$ canonically contains a subgp $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

Pf:
$$\begin{tikzcd} 0 \arrow[r] & \frac{1}{n}\mathbb{Z}/\mathbb{Z} \arrow[r, hook] \arrow[d, dashed, "\psi"', hook] & H^2(K^{ur}/K) \arrow[r, "Res"] \arrow[d, hook, "Inf"] \arrow[dr, phantom, "G"] & H^2(L^{ur}/L) \arrow[d, hook, "Inf"] \\ 0 \arrow[r] & H^2(L/K) \arrow[r, hook] & H^2(K^{sep}/K) \arrow[r, "Res"] \arrow[d, "Res"] & H^2(L^{sep}/L) \arrow[d, "Res"] \\ & & & \end{tikzcd}$$

inj by commutativity

Columns are inf-res exact sequences, so is the 2^{nd} row.

<u>Claim B.</u> ψ is an iso.

<u>Sublemma.</u> Let E/K be fin Gal, $G = Gal(E/K)$. Then there is $V \subseteq \mathcal{O}_E$ open G-submodule s.t. $\hat{H}^r(G,V)=0$ $\forall r \geq 0$.

Pf: Additive H90 $\Rightarrow$ $\hat{H}^r(G,(E,+)) = 0$ $\forall r$

Proof of AH90: $\exists$ normal basis for E/K, i.e. $\exists \alpha \in E$: $E = \langle \sigma\alpha \mid \sigma \in G \rangle$ as a K-vct. sp.

$\Rightarrow$ all elts in $\mathcal{O}_E$ are of the form $\frac{1}{N}\sum m_\sigma(\sigma\alpha)$, $m_\sigma \in \mathcal{O}_K$.

Multiply by N to kill the denominator $\Rightarrow$ get a normal basis for $N\cdot\mathcal{O}_E =: V$,

this is clearly a G-module by construction. Also open because quotient is finite.

$V \simeq \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathcal{O}_K$ induced $\Rightarrow$ has vanishing $\hat{H}^\bullet$

Sublemma 2. $\exists W \subseteq U_E$ (// $\mathcal{O}_E^\times$) open subgp which is a G-submodule and $H^r(G, W) = 0 \quad \forall r \geq 1$.

Pf: Use $\exp(T) = \sum_{n\geq 0} \frac{T^n}{n!}$ and $\log(T) = \sum_{n \geq 1} \frac{(T-1)^n}{n}$

These induce a homeo b/w open nbhds of the identity in L and $L^\times$.

Use $\pi_E^M V$ for π_E an unif of E and $M \gg 0$ s.t. $\pi_E^M V \xleftrightarrow{\cong}$ img of $\pi_E^M V =: W$.

Since Galois action commutes w/ addition & multiplication, exp and log respect the Galois action $\Rightarrow$ we indeed get a homeo and a G-module isomorphism.

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Lemma B. If L/K is cyclic $\Rightarrow h(\mathcal{O}_L^\times) = 1, \quad h(L^\times) = [L:K]$

Pf: $W \subseteq \mathcal{O}_L^\times$ open subgp, $\mathcal{O}_L^\times$ is compact $\rightarrow W \hookrightarrow \mathcal{O}_L^\times \twoheadrightarrow \mathcal{O}_L^\times/W$. the quotient is compact and has discrete topology. $\rightarrow$ finite. $\Rightarrow h(\mathcal{O}_L^\times/W) = 1$

$$h(\mathcal{O}_L^\times) = h(W) \cdot h(\mathcal{O}_L^\times/W) = h(W) = \frac{\# \hat{H}^0(G,W)}{\# \hat{H}^1(G,W)} = 1$$

$$\mathcal{O}_L^\times \longrightarrow L^\times \longrightarrow \mathbb{Z} \quad \Rightarrow \quad h(L^\times) = h(\mathcal{O}_L^\times) h(\mathbb{Z}) = 1 \cdot h(\mathbb{Z}) = \frac{\# \hat{H}^0(G,\mathbb{Z})}{\# \hat{H}^1(G,\mathbb{Z})} =$$

$$= \frac{\# \left| \frac{\mathbb{Z}^G}{\mathrm{Nm}_{L/K}(\mathbb{Z})} \right|}{\# \mathrm{Hom}(G, \mathbb{Z})} = \frac{\# \left| \frac{\mathbb{Z}}{[L:K]\mathbb{Z}} \right|}{1} = [L:K]$$

Prop. (Verifying (A2)) L/K fin Gal, $\Rightarrow H^2(L/K)$ cyclic of order $n := [L:K]$.

Pf. Sts $\# H^2(L/K) \leq n$: Fact from last time: $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \longleftrightarrow H^2(L/K)$, then use that G is a solvable group. $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$

Induction on $\#G$. $L/K'/K$ proper Galois subextension

$$0 \longrightarrow H^2(K'/K) \xrightarrow{\mathrm{Inf}} H^2(L/K) \xrightarrow{\mathrm{Res}} H^2(L/K') \longrightarrow \underbrace{\ldots}_{\text{may not terminate}^*}$$

$$\Rightarrow \# H^2(L/K) \leq \# H^2(K'/K) \cdot \# H^2(L/K')$$

both solvable, the induction hypothesis holds for them

$$\leq [K':K] \cdot [L:K'] = [L:K] = n.$$

(Btw from this it follows that the ses terminates, i.e. $* = 0$.)

$K/\mathbb{Q}_p$ fin extn

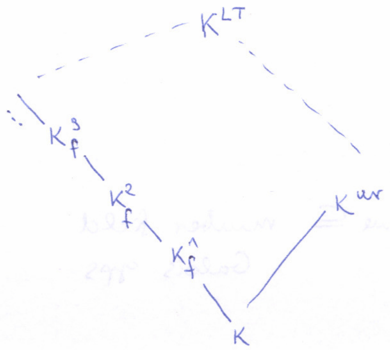
__Thm.__ There is a unique group hom $\mathrm{Art}_K : K^\times \longrightarrow \mathrm{Gal}(K^{ab}/K)$ s.t.

a) $\forall \pi_K$ uniformiser of K: $\forall K^{ab} \supseteq L \supseteq K$ ur extn: $\mathrm{Art}_K(\pi_K)|_L$ is the unique lift of the Frobenius.

b) $\forall L/K$ fin ab ext: $\mathrm{Nm}_{L/K}(L^\times) \subseteq \mathrm{Ker}(\mathrm{Art}_K)$, which gives an iso

$$K^\times / \mathrm{Nm}_{L/K}(L^\times) \xrightarrow{\sim} \mathrm{Gal}(L/K)$$

__Slogan.__ Reciprocity map: uniformisers $\longleftrightarrow$ Frobenius lifts.



$$\mathrm{Gal}(K^{LT}/K) \simeq \underbrace{\mathrm{Gal}(K^{ur}/K)}_{\substack{\wr\wr \\ \hat{\mathbb{Z}}}} \times \underbrace{\mathrm{Gal}\left(\bigcup_m K_f^m/K\right)}_{\substack{\wr\wr \\ \varprojlim_m \mathcal{O}_K/U_K^m \simeq \mathcal{O}_K^\times}}$$

$K^\times \xrightarrow{\mathrm{Art}_K} \mathrm{Gal}(K^{LT}/K)$, $K^{LT} = K^{ab}$,

$$\begin{tikzcd} L^\times \arrow[r, "\mathrm{Art}_L"] \arrow[d, "\mathrm{Nm}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Gal}(L^{ab}/L) \arrow[d, "\mathrm{res}"] \\ K^\times \arrow[r, "\mathrm{Art}_K"] & \mathrm{Gal}(K^{ab}/K) \end{tikzcd}$$

(we haven't proven these)

__Def.__ L/K alg extn, $N_{L/K}(L^\times) \subseteq K^\times$ norm group.

If L is a field having a prescribed subgp $N \subseteq K^\times$ as its norm gp, we call L the class field of N.

__Thm.__ (Existence Thm.) The norm groups in $K^\times$ are precisely the fin index subgps.

$$\mathbb{A}_K = \prod_v{}' (K_v; \mathcal{O}_v) \quad \text{where } v \text{ runs on the places}$$

$$= \underbrace{\prod_{\mathfrak{p}}{}' (K_v; \mathcal{O}_v)}_{} \oplus \prod_{\sigma} K_v \quad \text{where } K_v \text{ is the completion wrt } v,\ \mathcal{O}_v \subseteq K_v \text{ the val. rg.}$$

$$= \left\{ (x_v)_v \in \prod_v K_v \;\middle|\; x_v \in \mathcal{O}_v \text{ for all but fin many } v \right\}$$

The adèle ring is a loc cpt top ring.

$\mathbb{I}_K := \mathbb{A}_K^\times = \prod_v{}' (K_v^\times : \mathcal{O}_v^\times)$ with __not__ the induced topology

__Def.__ $K^\times \xrightarrow{i} \mathbb{I}_K$, $x \longmapsto (x, x, \ldots)$. diagonal.

Idèle class group: $C_K := \mathbb{I}_K / i(K^\times)$

Global Input Theorem. K number field, L/K fin Galois, $G = \mathrm{Gal}(L/K)$, $M := C_L$. Then these satisfy (A1) and (A2) in the Tate–Nakayama theorem:

(A1) $H^1(H, C_L) = 0 \quad \forall H \leq G \quad \longrightarrow$ 'Artin–Tate theorem'

(A2) $H^2(H, C_L)$ is cyclic of order $\#$ $\longrightarrow$ "proving the First and Second Inequality"

Thm. There is a canonical gp hom $\mathrm{Art}_K^{\text{global}} : \mathbb{I}_K \longrightarrow \mathrm{Gal}(K^{ab}/K)$ s.t.

- $\mathrm{Art}_K^{\text{global}}(i(K^\times)) = 0$,
- $\forall L/K$ fin ab extn: $\mathrm{Art}_K^{\text{global}} : C_K / \mathrm{Nm}_{L/K}(C_L) \xrightarrow{\sim} \mathrm{Gal}(L/K)$
- $\forall L/K$ fin ab extn: $\mathrm{Art}_K^{\text{global}} = \prod_v \underbrace{\mathrm{Art}_{K_v}}_{\text{local reciprocity map}}$

Thm. $\underbrace{D_{-/p}}_{\text{decomp gp.}} \xrightarrow{\sim} \mathrm{Gal}(K_p^{ab}/K_p)$

decomp gp at a prime $\subseteq$ number field Galois gps

$\|$

Gal gp of completion at a prime